

Complex Analysis Preliminary Exam

August 4, 2023

Justification should be provided for all responses.

1. For what values of $a, b \in \mathbb{R}$ is $f(x + iy) = e^{ax} \cos(by)$ for $x, y \in \mathbb{R}$ the real part of a holomorphic function?
2. Let $U \subseteq \mathbb{C}$ be a nonempty open set, and let $g : U \rightarrow \mathbb{C}$ be a bounded holomorphic function. Let $K = \sup\{\operatorname{Im}(g(z)) : z \in U\}$. Under what conditions does there exist $z_0 \in U$ such that $\operatorname{Im}(g(z_0)) = K$?
3. Find the set of all holomorphic functions h on \mathbb{C} such that $|h'(z) + 2023| \leq |z|$ for all $z \in \mathbb{C}$.
4. By using a complex contour integral, find

$$\int_{-\infty}^{\infty} \frac{\sin(x) + \cos(2x)}{1 + x^2} dx.$$

5. Find the largest open set in \mathbb{C} where the sum

$$f(z) = \sum_{m=0}^{\infty} \frac{e^{-mz}}{m+1}$$

converges to an analytic function. Can f be analytically continued to a larger set?

6. Let $F(z) = e^z - 5z^2$ for $z \in \mathbb{C}$. How many values of z are fixed by the map $F : \Delta \rightarrow \mathbb{C}$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$?
7. Find a conformal map $\phi : S \rightarrow H$, where $S = \{x + iy : x, y \in \mathbb{R}, x^2 + y^2 < 4, y < 0\}$ and $H = \{x + iy : x, y \in \mathbb{R}, x < 0\}$, such that $\phi(-i) = -2 + 3i$. Is the ϕ you found the only possible solution?
8. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let $\beta : \Delta \rightarrow \Delta$ be a holomorphic function such that $\beta(\frac{2}{3}) = 0$. Find all possible values of $|\beta(\frac{9}{11})|$.